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## LETTER TO THE EDITOR

# Storage of correlated patterns in a perceptron

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**Abstract.** We calculate the storage capacity of a perceptron for correlated Gaussian patterns. We find that the storage capacity  $\alpha_c$  can be less than 2 if similar patterns are mapped onto different outputs and *vice versa*. As long as the patterns are in a general position we obtain, in contrast to previous works, that  $\alpha_c \geq 1$  in agreement with Cover's theorem. Numerical simulations confirm the results.

The critical storage capacity of a simple perceptron for randomly chosen input/output pairs is known to be  $\alpha_c = p/N = 2$ , with  $p$  the number of stored patterns and  $N$  their input dimension. This result was first derived by Cover (1965) using a geometrical argument and later by Gardner (1988) and Gardner and Derrida (1988) by calculating the fractional phase space volume of consistent couplings with the tools of statistical mechanics and the replica approach.

Cover's theorem states that, as long as the patterns  $\xi^\mu$  ( $\mu = 1, \dots, p$ ) are in a general position (no subset of  $N$  or less patterns is linear dependent), the critical storage capacity  $\alpha_c$  is at least 1, independently of the corresponding outputs  $s^\mu$ . For  $\alpha$  between 1 and 2 the fraction of output combinations which are not linearly separable is exponentially small in  $N$ . The converse holds for  $\alpha$  larger than 2. In this case a randomly chosen sequence of outputs will not be linearly separable with a probability approaching 1 as  $N \rightarrow \infty$ . This is even true if correlations among the patterns  $\xi^\mu$  are introduced (Monasson 1992).

One would argue that, in general, correlations which include the outputs lead to higher critical storage capacities as is, for instance, the case for biased patterns (Gardner 1988). Recently it has been found (Bork 1994, Schröder *et al* 1995) that patterns and outputs extracted from a bit-sequence seem to lead to smaller storage capacities than  $\alpha_c = 2$  for a perceptron. The bit-sequence is thought of as an infinite time series  $\zeta_i$  ( $i = 1, 2, \dots$ ) in which the first pattern results from the first  $N$  values  $\zeta_i$  ( $i = 1, \dots, N$ ) and its output from the  $(N+1)$ th value  $\zeta_{N+1}$  (or  $\text{sign}(\zeta_{N+1})$  for continuous valued  $\zeta_i$ ). By moving this  $(N+1)$ -broad window one step forward, the second pattern and its corresponding output result, and so on. If the  $\zeta_i$  are drawn at random from a distribution with  $\langle \zeta_i \rangle = 0$ ,  $\langle \zeta_i^2 \rangle = 1$ , one finds that the critical storage capacity of a perceptron which stores the resulting input/output pairs  $(\xi^\mu, s^\mu)$  is  $\alpha_c \approx 1.82$  for binary  $\zeta_i$  and  $\alpha_c \approx 1.88$  for Gaussian  $\zeta_i$ . This result indicates that the embedded correlations between input/output pairs resulting from the bit-sequence lead to a reduction of the storage capacity when compared to randomly chosen pairs and are harder to implement in a perceptron.

Taking the former as a motivation, we will investigate in the present letter the effect of correlations between input/output pairs on the critical storage capacity of a perceptron. The main idea is that the storage of similar patterns with different outputs should be more difficult to implement in a perceptron than the case of similar patterns with identical outputs.

If we introduce the transformed patterns  $\sigma_\mu = \xi_\mu s^\mu$ , this similarity or dissimilarity can be described with a positive or negative overlap ( $R = N^{-1} \sigma_\mu \cdot \sigma_\nu$ ) respectively between two transformed patterns  $\mu$  and  $\nu$ . Without loss of generality we fix all outputs to have the value  $s^\mu = +1$  ( $\mu = 1, \dots, p$ ), so that  $\sigma_\mu = \xi_\mu$ . Let us now take pairs of patterns with a fixed overlap  $R$ . With a normalization such that  $\xi_\mu^2 = N$ , we have

$$\xi_{2\mu-1} \cdot \xi_{2\mu} = NR \quad \text{for } \mu = 1, \dots, p/2 \quad (1)$$

with  $|R| \leq 1$ . Apart from these fixed overlaps two arbitrary patterns will not be correlated and thus will have an overlap of  $N^{-1} \xi_\mu \cdot \xi_\nu = 0$  in the thermodynamic limit  $N \rightarrow \infty$ .

Before evaluating the storage capacity for a general  $R$  let us consider some special cases of interest. For  $R = 1$  the two patterns out of a pair are identical (for  $N \rightarrow \infty$ ), so that the storage of the first pattern automatically implements the second one, concluding that  $\alpha_c(R = 1) = 4$ . For  $R = 0$  no correlations are present and  $\alpha_c(R = 0) = 2$  as usual. For  $R = -1$  the patterns are not in a general position, since  $p/2$  of them are pairwise linearly dependent. Already the first two patterns cannot be implemented by a perceptron, so that  $\alpha_c(R = -1) = 0$ . However, if  $R$  is very close to  $-1$  the general position is guaranteed and thus  $\alpha_c(R = -1 + \epsilon) \geq 1$ , ( $\epsilon > 0$ ) according to Cover's theorem. We will investigate this case later in more detail.

Following Gardner's approach we consider the fractional phase space volume  $V$  of couplings  $J$  that are consistent with the constraints imposed by the patterns (version space):

$$V = V_{\text{tot}}^{-1} \int \prod_{i=1}^N dJ_i \prod_{\mu=1}^p \Theta \left( s^\mu \frac{1}{\sqrt{N}} \sum_{i=1}^N J_i \xi_i^\mu - \kappa \right) \delta \left( \sum_{i=1}^N J_i^2 - N \right) \quad (2)$$

with

$$V_{\text{tot}} = \int \prod_{i=1}^N dJ_i \delta \left( \sum_{i=1}^N J_i^2 - N \right). \quad (3)$$

It is clear that, for negative  $R$ , the reduction in the version space by every new pair of patterns is more drastic than for  $R = 0$  and so we expect  $\alpha_c$  to be less than 2.

We now fix  $s^\mu = +1$  ( $\mu = 1, \dots, p$ ) and introduce the conditions (1) via delta functions. The average  $\langle \langle \ln V \rangle \rangle_\xi$  will be performed by means of the replica trick and is defined by

$$\langle \langle f(\{\xi^\nu\}_{\nu=1}^p) \rangle \rangle_\xi = C_N \int \prod_{\mu=1}^{p/2} \prod_{i=1}^N (D\xi_i^{2\mu-1} D\xi_i^{2\mu}) f(\{\xi^\nu\}_{\nu=1}^p) \prod_{\mu=1}^{p/2} \delta(N^{-1} \xi_{2\mu-1} \cdot \xi_{2\mu} - R) \quad (4)$$

where  $C_N$  is the normalization constant resulting from  $\langle \langle 1 \rangle \rangle_\xi = 1$ , and  $Dx \equiv dx \exp(-x^2/2)/\sqrt{2\pi}$  is the Gaussian measure. As usual in the calculation the order parameter  $q_{\alpha\beta}$  appears, which measures the overlap between two replicas. Making the replica symmetric ansatz  $q_{\alpha\beta} = q$  ( $\forall \alpha < \beta$ ) and using  $\langle \langle \ln V \rangle \rangle_\xi = \lim_{n \rightarrow 0} n^{-1} \ln \langle \langle V^n \rangle \rangle_\xi$  together with the saddle-point method to evaluate the integrals in the thermodynamic limit, one obtains

$$\frac{1}{N} \langle \langle \ln V \rangle \rangle_\xi = \text{Extr}_q \left\{ \frac{\alpha}{2} \int Dx \int Dz \ln f(q, R, \kappa, x, z) + \frac{1}{2} \ln(1-q) + \frac{1}{2} \frac{q}{1-q} \right\} \quad (5)$$

with

$$f(q, R, \kappa, x, z) = \sqrt{1-R^2} \int_\gamma^\infty D\omega \int_\delta^\infty Du \exp(Ru\omega) \quad (6)$$

where

$$\gamma = \frac{\kappa + \sqrt{q}x}{\sqrt{1-q}\sqrt{1-R^2}} \quad \delta = \frac{\kappa + \sqrt{q}(Rx + \sqrt{1-R^2}z)}{\sqrt{1-q}\sqrt{1-R^2}} \quad (7)$$

The critical storage capacity is reached when the version space shrinks to a single point and thus  $q$  reaches unity. From the extremum condition in (5) we obtain for the critical storage capacity  $\alpha_c$ :

$$\alpha_c^{-1}(R, \kappa) = \lim_{q \rightarrow 1} \left[ - (1 - q)^2 \int Dx \int Dz \frac{\partial}{\partial q} \ln f(q, R, \kappa, x, z) \right]. \quad (8)$$

In the limit  $q \rightarrow 1$  one has to consider the cases of positive and negative  $R$  separately yielding

$$\begin{aligned} \alpha_c^{-1}(R, \kappa) &= \int_{-\infty}^{x_0} Dx \int_{z_0}^{\infty} Dz \frac{b^2}{2} + \int_{x_0}^{\infty} Dx \int_{-\infty}^{z_1} Dz \frac{a^2}{2} \\ &\quad + \int_{x_0}^{\infty} Dx \int_{z_1}^{z_2} Dz \frac{a^2 - 2Rab + b^2}{2(1 - R^2)} + \int_{x_0}^{\infty} Dx \int_{z_2}^{\infty} Dz \frac{b^2}{2} \quad (R > 0) \\ \alpha_c^{-1}(R, \kappa) &= \int_{-\infty}^{x_0} Dx \int_{z_0}^{z_2} Dz \frac{b^2}{2} + \int_{-\infty}^{x_0} Dx \int_{-\infty}^{z_2} Dz \frac{a^2 - 2Rab + b^2}{2(1 - R^2)} \\ &\quad + \int_{x_0}^{\infty} Dx \int_{-\infty}^{z_1} Dz \frac{a^2}{2} + \int_{x_0}^{\infty} Dx \int_{z_1}^{\infty} Dz \frac{a^2 - 2Rab + b^2}{2(1 - R^2)} \quad (R < 0) \end{aligned}$$

with  $a = \kappa + x$ ,  $b = \kappa + Rx + \sqrt{1 - R^2}z$  and

$$z_0 = -\frac{\kappa + Rx}{\sqrt{1 - R^2}} \quad z_1 = -\frac{\kappa(1 - R)}{\sqrt{1 - R^2}} \quad z_2 = \frac{\kappa(1 - R) + x(1 - R^2)}{R\sqrt{1 - R^2}}. \quad (9)$$

For the interesting case  $\kappa = 0$  this expression reduces to a surprisingly simple form:

$$\alpha_c^{-1}(R, \kappa = 0) = \left( \frac{1}{4} + \frac{\phi}{2\pi} \right) \quad \phi = \arccos R. \quad (10)$$

Here  $\phi$  is the angle between two correlated patterns, and (10) holds for  $-1 < R \leq 1$ . The resulting curve is plotted in figure 1. The simulations have been carried out in the following two ways. First, one can calculate the average probability that a given set of patterns with correlations as described above is linearly separable for different values of  $\alpha$ . The condition for the critical capacity is that this probability equals 1/2. The second method is to assume that the median learning time (for the perceptron learning rule) scales as  $\tau^{-0.5} \sim (\alpha_c - \alpha)$  for  $\alpha \rightarrow \alpha_c$ . This can be proven for uncorrelated patterns (Oppen 1988) and has been used by Priel *et al* (1994) to determine the critical capacity by extrapolating the curve to  $\tau \rightarrow \infty$ . The inset in figure 1 shows clearly that the mentioned scaling law is obeyed in our case too.

As we expected, for negative  $R$  the storage capacity lies below 2, approaching the value of  $\alpha_c = 4/3$  for  $R \rightarrow -1$ . This result can be understood as follows. Every pair  $\mu = 1, \dots, p/2$  of correlated patterns defines a vector  $L_\mu = \xi_{2\mu-1} - \xi_{2\mu}$ . For  $R \rightarrow -1$  the coupling vector  $J$  falls for every  $\mu$  into a hyperhalfplane orthogonal to  $L_\mu$ . This leads to the constraints:

$$\begin{aligned} (1) \quad L_\mu \cdot J &\rightarrow 0 \\ (2) \quad T_\mu \cdot J &\geq 0 \quad \mu = 1, \dots, p/2 \end{aligned} \quad (11)$$

for  $J$ , where  $T_\mu$  is a vector orthogonal to  $L_\mu$ . The second constraint defines a new perceptron problem with  $p/2$  uncorrelated patterns  $T_\mu$  for a coupling vector  $J$  with an effective number of dimensions of  $(N - p/2)$  due to the first constraint. According to Cover's theorem  $(p/2)/(N - p/2) = 2$  and thus  $\alpha_c = p/N = 4/3$ .

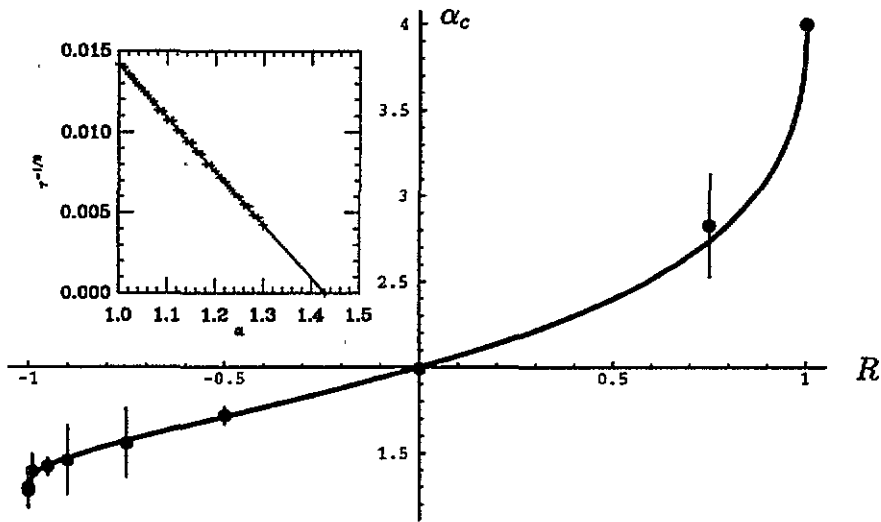


Figure 1. The critical storage capacity  $\alpha_c$  as a function of the overlap  $R$  between two correlated patterns for  $\kappa = 0$ . The dots with their corresponding error bars are results from numerical simulations for systems with  $N = 100$ . Inset: The median learning time to the power of  $-1/2$  as a function of  $\alpha$  for  $R = -0.95$ . The values are averaged over 1000 samples, the line is a least square fit for the data and the intersection between the extrapolation and the  $x$  axis gives the estimated value of  $\alpha_c$ .

Up to now we have assumed that the overlap  $R$  is the same for all pairs. The case where two patterns have an overlap  $R$  with a probability  $P(R)$  can also be handled. The averages over different values of  $R$  factorize and the storage capacity is simply given by

$$\alpha_c^{-1}(\kappa = 0) = \int_{-\infty}^{\infty} dR P(R) \left( \frac{1}{4} + \frac{\phi}{2\pi} \right) = \int_{-\infty}^{\infty} dR P(R) \alpha_c^{-1}(R, \kappa = 0). \quad (12)$$

The last equality states that the reciprocal values of the capacity for a fixed value of the overlap weighted with their corresponding probability sum up to the reciprocal of the total capacity. If the distribution is symmetric ( $P(R) = P(-R)$ ) the storage capacity is  $\alpha_c = 2$ . This is immediately clear if one thinks of a correlation with a primary distribution  $P_1(R)$  and, in addition, chooses outputs  $s^\mu$  at random  $\pm 1$  instead of taking them all equal to  $+1$ . The new distribution of the correlations is then given by  $P(R) = P_1(R)/2 + P_1(-R)/2$ , which is symmetric and will lead in (12) to  $\alpha_c = 2$  as it should, since we have taken the outputs to be random.

Let us now turn to the case where more than two patterns are correlated among each other. In the case of three patterns with equal pairwise overlap  $R$  we can conclude that for  $R = 1$  the storage capacity is  $\alpha_c = 6$  for  $\kappa = 0$ . If  $R$  tends to  $-1/2$  which is the minimal accessible value in this case, a geometrical argument similar to (11) leads us to  $(p/3)/(N - 2p/3) = 2$  and thus  $\alpha_c = p/N = 6/5$ . The calculation of  $\alpha_c(R)$  for other values of  $R$  should be more complicated than in the former case, since to disentangle the additional correlations one has to introduce more Gaussian fields. In general, if the patterns are correlated in tuples of  $m$  it follows in the same way:

$$\alpha_c(R = 1, \kappa = 0) = 2m$$

$$\alpha_c(R = 0, \kappa = 0) = 2$$

$$\alpha_c \left( R \rightarrow \frac{-1}{m-1}, \kappa = 0 \right) = \frac{2m}{2m-1}. \tag{13}$$

For  $m \rightarrow \infty$ ,  $\alpha_c$  tends to 1 as  $R$  approaches the minimal possible value  $-1/(m-1)$ .

As long as  $m$  is of the order 1 compared to the number of patterns  $p$ , we will have to calculate  $\alpha_c(R)$  as for  $m = 2$ , which for larger  $m$  becomes very complicated. If  $m$ , however, is of the order  $p^\gamma$  with  $0 < \gamma \leq 1$  we are able to proceed in a much simpler way. For this purpose we first introduce, as in Fontanari and Meir (1989), the correlation matrix  $C$  for the patterns:

$$c_{\mu\nu} = \langle \xi_i^\mu \xi_i^\nu \rangle_\xi \quad \forall \mu, \nu = 1, \dots, p. \tag{14}$$

The  $\xi_i = (\xi_i^1, \dots, \xi_i^p)$  are then distributed as

$$P(\xi_i) = \frac{1}{\sqrt{(2\pi)^p \det C}} \exp \left( -\frac{1}{2} \xi_i^T C^{-1} \xi_i \right). \tag{15}$$

The class of correlations we have considered is described by

$$c_{\mu\nu} = \begin{cases} R & \text{for } \mu = m\sigma - \tau, \nu = m\sigma - \epsilon, \sigma = 1, \dots, p/m \\ & \text{with } \tau, \epsilon = 0, \dots, (m-1) \quad (\tau \neq \epsilon) \\ 1 & \text{for } \mu = \nu \\ 0 & \text{else.} \end{cases} \tag{16}$$

The average over the fractional phase space of couplings is now performed with (15) and results for  $m \rightarrow \infty$  and  $N \rightarrow \infty$  in

$$\frac{1}{N} \langle \ln V \rangle_\xi = \text{Extr}_{q, \hat{r}} \left\{ \alpha \text{Extr}_{r, \hat{r}} \left\{ -ir\hat{r} - \frac{1}{2}c(1-q)r^2 + \int Dt \ln H(\omega) \right\} + \frac{1}{2} \ln(1-q) + \frac{1}{2} \frac{q}{1-q} \right\} \tag{17}$$

with

$$\omega = \frac{\kappa + \hat{r} + \sqrt{q}t}{\sqrt{1-q}} \quad H(x) = \int_x^\infty Dt \tag{18}$$

and  $c = Rm$  in the limit  $m \rightarrow \infty$ , so  $-1 < c \leq \infty$  since  $-(m-1)^{-1} < R \leq 1$ . We have made the replica symmetric ansatz for  $q_{\alpha\beta}$  and the additional order parameter  $r_\alpha = m^{-1} \sum_{\mu=1}^m x_\alpha^\mu$ , where  $x_\alpha^\mu$  are the conjugate variables to the local fields  $\lambda_\alpha^\mu$ , and its conjugate  $\hat{r}_\alpha$ .

If we solve (17) for the extremum we first can write  $r$  in terms of  $\hat{r}$  and find in the limit  $q \rightarrow 1$ :

$$-\frac{\hat{r}}{c} = (\kappa + \hat{r})H(-\kappa - \hat{r}) + \frac{1}{\sqrt{2\pi}} \exp \left\{ -\frac{1}{2}(\kappa + \hat{r})^2 \right\} \tag{19}$$

$$\alpha_c^{-1}(c, \kappa) = H(-\kappa - \hat{r}) + \kappa \frac{\hat{r}}{c}. \tag{20}$$

From the first equation we can obtain  $\hat{r}$  numerically and plug it into the second to find  $\alpha_c$ . Figure 2 shows the storage capacity as a function of  $c$  for several values of  $\kappa$ . For  $\kappa = 0$ ,  $\alpha_c$  approaches 1 as  $c \rightarrow -1$ . This behaviour is in agreement with (13) for large  $m$ . Fontanari and Meir (1989) considered the case  $m = p$ , so all patterns are pairwise correlated. However, due to an error in one of the saddle-point equations  $\alpha_c(\kappa = 0)$  becomes less than 1 for values below  $c \approx -0.7$  and reaches  $\alpha_c = 0$  for  $c \rightarrow -1$ .

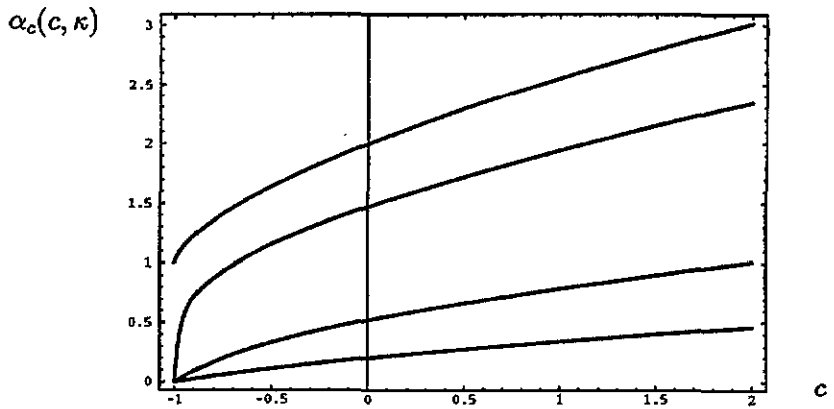


Figure 2. The critical storage capacity  $\alpha_c$  in the case where the patterns are correlated in tuples of  $m$  with a pairwise overlap  $R$ , as a function of the parameter  $c = Rm$  for  $m \rightarrow \infty$  and for various values of  $\kappa = 0.0, 0.2, 1.0, 2.0$  (from top to bottom).

For hierarchically correlated biased patterns the storage capacity has been calculated by Engel (1990). Above certain values of the bias the critical capacity is less than two and for certain ranges even less than one. It is not clear to us at which point the general position is violated in this case.

In summary, we have analysed the behaviour of the storage capacity of a perceptron for correlated patterns. We find that the storage capacity is lowered with respect to uncorrelated patterns when different patterns (negative overlap) are mapped onto the same output, but does not fall below one, in agreement with Cover's theorem. As a consequence we suggest that the correlation matrix of the patterns should be analysed for problems which lead to a reduced storage capacity as, for example, the bit sequence.

Future work should include the calculation of the stability of the replica symmetric solution and the storage capacity for a binary perceptron. One could also investigate the consequences for other architectures such as the committee or parity machine, although we think that the results should be similar for these cases.

After completion of this work we have learned that a similar problem has been studied by Winkel (1995) using a different approach.

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